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Exact Solitary Wave Solutions to a Class of Generalized Odd-Order KdV Equations

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Using a proper ansatz, we have obtained a series of exact solitary wave solutions to a class of generalized odd-order KdV equations.

The generalized odd-order KdV equation of the form

$$u_t + \beta_1 u^\alpha u_x + \sum_{n=2}^{m} \beta_n u_{(2n-1)x} = 0 \tag{1}$$

where β_i ($i = 1, 2, 3, \ldots, m$) are real numbers, has been widely used in the physical sciences (Johnson, 1980; Drazin and Johnson, 1989; Sachdev, 1987; Newell and Moloney, 1992; Kakutari and Ono, 1969; Kawahara, 1972; Yoshimura and Watanabe, 1982; Dai, 1982; Hereman *et al.*, 1986; Hereman and Takaoka, 1990; Hooper and Grimshaw, 1988; Ma, 1993). Looking for traveling (and/or solitary) wave solutions to this equation has been an important topic for several decades. By introducing an ansatz equation, we have obtained a new class of solitary wave solutions to this KdV equation. These results and examples of possible applications are presented in this paper.

For the traveling wave solution to the above KdV equation the variable transformation

$$\xi = x - ct \tag{2}$$

where c is the speed of the traveling *wave*, may be used. Thus, equation (1) is transformed to

$$-cu' + \beta_1 u^\alpha u' + \sum_{n=2}^{m} \beta_n u^{(2n-1)} = 0 \tag{3}$$

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Let us consider the following ansatz (Yang, 1994):

$$\frac{du}{d\xi} = u' = -\nu b u\left[1 - \left(\frac{u}{a}\right)^{2/\nu}\right]^{1/2} \tag{4}$$

where ν, a, and b are real numbers, and $\nu > 0$. Through integration, we obtain the solution as

$$u(\xi) = a\,\mathrm{sech}^{\nu}(b\xi + c_0) \tag{5}$$

Thus, we have

$$u'' = \nu b^2 u\left[\nu - (\nu + 1)\left(\frac{u}{a}\right)^{2/\nu}\right]$$

$$= \frac{-b}{[1 - (u/a)^{2/\nu}]^{1/2}}\left[\nu - (\nu + 1)\left(\frac{u}{a}\right)^{2/\nu}\right]u'$$

$$u''' = b^2\left[\nu^2 - (\nu + 1)(\nu + 2)\left(\frac{u}{a}\right)^{2/\nu}\right]u'$$

$$u^{(4)} = \frac{-b^3}{[1 - (u/a)^{2/\nu}]^{1/2}}\left[\nu^3 - 2(\nu + 1)(2 + 2\nu + \nu^2)\left(\frac{u}{a}\right)^{2/\nu} + (\nu + 1)(\nu + 2)(\nu + 3)\left(\frac{u}{a}\right)^{4/\nu}\right]u'$$

$$u^{(5)} = b^4\left[\nu^4 - 2(\nu + 1)(\nu + 2)(2 + 2\nu + \nu^2)\left(\frac{u}{a}\right)^{2/\nu} + (\nu + 1)(\nu + 2)(\nu + 3)(\nu + 4)\left(\frac{u}{a}\right)^{4/\nu}\right]u'$$

$$u^{(6)} = \frac{-b^5}{[1 - (u/a)^{2/\nu}]^{1/2}}\left[\nu^5 - (\nu + 1)(4 + 2\nu + \nu^2)(4 + 6\nu + 3\nu^2)\left(\frac{u}{a}\right)^{2/\nu} + (\nu + 1)(\nu + 2)(\nu + 3)(20 + 12\nu + 3\nu^2)\left(\frac{u}{a}\right)^{4/\nu} - (\nu + 1)(\nu + 2)(\nu + 3)(\nu + 4)(\nu + 5)\left(\frac{u}{a}\right)^{6/\nu}\right]u'$$

$$u^{(7)} = b^6\left[\nu^6 - (\nu + 1)(\nu + 2)(4 + 2\nu + \nu^2)(4 + 6\nu + 3\nu^2)\left(\frac{u}{a}\right)^{2/\nu}\right.$$

$$+ (\nu + 1)(\nu + 2)(\nu + 3)(\nu + 4)(20 + 12\nu + 3\nu^2)\left(\frac{u}{a}\right)^{4/\nu}$$

$$\left.- (\nu + 1)(\nu + 2)(\nu + 3)(\nu + 4)(\nu + 5)(\nu + 6)\left(\frac{u}{a}\right)^{6/\nu}\right]u'$$

$$\vdots$$

It is easy to see that all high odd-order (≥ 3) derivatives of u can be expressed as a product of u' and a (pseudo-)polynomial of $u^{2/\nu}$, i.e., in the form

$$u^{(2n-1)} = f_n(u)u', \qquad n = 1, 2, 3, \ldots, m \tag{6}$$

where $f_n(u)$ are the $(n - 1)$th-order (pseudo-)polynomials of $u^{2/\nu}$. Substituting these expressions into equation (3), we have the equality

$$-c + \beta_1 u^\alpha + \sum_{n=2}^{m} \beta_n f_n(u) \equiv 0 \tag{7}$$

Setting all of the coefficients of the $u^{2/\nu}$ polynomials to zero, we can determine the values of a, b, and ν (as well as c for $n \geq 3$), and the relationships between the β_i ($i = 1, 2, 3, \ldots, m$) for $n \geq 4$, under which equation (1) has the solitary wave solution described by equation (5). We now consider some possible examples of these results for different values of m.

Example 1. For $m = 2$, equation (7) can be written explicitly as

$$\beta_1 u^\alpha - c + \beta_2 b^2\left[\nu^2 - (\nu + 1)(\nu + 2)\left(\frac{u}{a}\right)^{2/\nu}\right] \equiv 0 \tag{8}$$

Let $\nu = 2/\alpha$; then we have the system of equations

$$c - \beta_2 b^2 \nu^2 = 0$$

$$\beta_1 - \beta_2 b^2(\nu + 1)(\nu + 2)a^{-\alpha} = 0$$

The solutions to these equations are

$$a = \left[\frac{c(\alpha + 1)(\alpha + 2)}{2\beta_1}\right]^{1/\alpha}$$

$$b = \pm\left(\frac{c\alpha^2}{4\beta_2}\right)^{1/2}$$

Example 2. For $m = 3$, equation (7) can be expressed explicitly as

$$\beta_1 u^\alpha - c + \beta_2 b^2\left[\nu^2 - (\nu + 1)(\nu + 2)\left(\frac{u}{a}\right)^{2/\nu}\right]$$
$$+ \beta_3 b^4\left[\nu^4 - 2(\nu + 1)(\nu + 2)(2 + 2\nu + \nu^2)\left(\frac{u}{a}\right)^{2/\nu}\right.$$
$$\left. + (\nu + 1)(\nu + 2)(\nu + 3)(\nu + 4)\left(\frac{u}{a}\right)^{4/\nu}\right] \equiv 0 \tag{9}$$

Let $\nu = 4/\alpha$; then we have the system of equations

$$c - \beta_2 b^2 \nu^2 - \beta_3 b^4 \nu^4 = 0$$
$$\beta_2 + 2\beta_3 b^2(2 + 2\nu + \nu^2) = 0$$
$$\beta_1 + \beta_3 b^4(\nu + 1)(\nu + 2)(\nu + 3)(\nu + 4)a^{-\alpha} = 0$$

The solutions are

$$a = \left[\frac{-\beta_2^2(\alpha + 1)(\alpha + 2)(3\alpha + 4)(\alpha + 4)}{2\beta_1\beta_3(8 + 4\alpha + \alpha^2)^2}\right]^{1/\alpha}$$
$$b = \pm\left[\frac{-\alpha^2\beta^2}{4\beta_3(8 + 4\alpha + \alpha^2)}\right]^{1/2}$$
$$c = \frac{-4\beta_2^2(\alpha + 2)^2}{\beta_3(8 + 4\alpha + \alpha^2)^2}$$

Example 3. For $m = 4$, equation (7) has the form

$$\beta_1 u^\alpha - c + \beta_2 b^2\left[\nu^2 - (\nu + 1)(\nu + 2)\left(\frac{u}{a}\right)^{2/\nu}\right]$$
$$+ \beta_3 b^4\left[\nu^4 - 2(\nu + 1)(\nu + 2)(2 + 2\nu + \nu^2)\left(\frac{u}{a}\right)^{2/\nu}\right.$$
$$\left. + (\nu + 1)(\nu + 2)(\nu + 3)(\nu + 4)\left(\frac{u}{a}\right)^{4/\nu}\right]$$
$$+ \beta_4 b^6\left[\nu^6 - (\nu + 1)(\nu + 2)(4 + 2\nu + \nu^2)(4 + 6\nu + 3\nu^2)\left(\frac{u}{a}\right)^{2/\nu}\right.$$

$$+ (\nu + 1)(\nu + 2)(\nu + 3)(\nu + 4)(20 + 12\nu + 3\nu^2)\left(\frac{u}{a}\right)^{4/\nu}$$

$$- (\nu + 1)(\nu + 2)(\nu + 3)(\nu + 4)(\nu + 5)(\nu + 6)\left(\frac{u}{a}\right)^{6/\nu}\Bigg] \equiv 0 \qquad (10)$$

Let $\nu = 6/\alpha$; then we have the system of equations;

$$c - \beta_2 b^2 \nu^2 - \beta_3 b^4 \nu^4 - \beta_4 b^6 \nu^6 = 0$$

$$\beta_2 + 2\beta_3 b^2(2 + 2\nu + \nu^2) + \beta_4 b^4(4 + 2\nu + \nu^2)(4 + 6\nu + 3\nu^2) = 0$$

$$\beta_3 + \beta_4 b^2(20 + 12\nu + 3\nu^2) = 0$$

$$\beta_1 - \beta_4 b^6(\nu + 1)(\nu + 2)(\nu + 3)(\nu + 4)(\nu + 5)(\nu + 6)a^{-\alpha} = 0$$

Under the condition

$$\beta_2\beta_4(27 + 18\alpha + 5\alpha^2)^2 = \beta_3^2(243 + 324\alpha + 162\alpha^2 + 36\alpha^3 + 4\alpha^4)$$

we obtain the solution as

$$a = \left[\frac{-9\beta_3^3(\alpha + 1)(\alpha + 2)(\alpha + 3)(2\alpha + 3)(5\alpha + 6)(\alpha + 6)}{8\beta_1\beta_4^2(27 + 18\alpha + 5\alpha^2)^3}\right]^{1/\alpha}$$

$$b = \pm\left[\frac{-\alpha^2\beta_3}{4\beta_4(27 + 18\alpha + 5\alpha^2)}\right]^{1/2}$$

$$c = \frac{-9\beta_3^3(9 + 9\alpha + 2\alpha^2)^2}{\beta_4^2(27 + 18\alpha + 5\alpha^2)^3}$$

This is a general solution to the seventh-order KdV equation for an arbitrary real number α [the result for $\alpha = 1$ was recently obtained by Ma (1993)].

Now, we present the "general" solutions to equation (3) for any given m = integer. Letting $\nu = 2(m - 1)/\alpha$, through some calculation we obtain the equation system

$$c - \sum_{n=2}^{m} \beta_n b^{2(n-1)}\nu^{2(n-1)} = 0$$

$$\sum_{n=2}^{m} \beta_n b^{2(n-2)}\delta_{n-2}(\nu) = 0$$

$$\sum_{n=3}^{m} \beta_n b^{2(n-3)}\xi_{n-3}(\nu) = 0$$

$$\sum_{n=4}^{m} \beta_n b^{2(n-4)} \eta_{n-4}(\nu) = 0$$

$$\sum_{n=5}^{m} \beta_n b^{2(n-5)} \zeta_{n-5}(\nu) = 0$$

$$\vdots$$

$$\beta_1 - \beta_m b^{2(m-1)} a^{-\alpha} \prod_{n=1}^{2(m-1)} (\nu + n) = 0$$

where the $\{\delta\}$, $\{\xi\}$, $\{\eta\}$, and $\{\zeta\}$, etc., are defined by the recursion formulas

$$\begin{aligned}
&\delta_0 = \xi_0 = \eta_0 = \zeta_0 = \cdots = 1 \\
&\delta_{n+1}(\nu) = \nu^{2(n+1)} + (\nu + 2)^2 \delta_n, && n = 0, 1, 2, \ldots, m - 1 \\
&\xi_{n+1}(\nu) = \delta_{n+1} + (\nu + 4)^2 \xi_n, && n = 0, 1, 2, \ldots, m - 2 \\
&\eta_{n+1}(\nu) = \xi_{n+1} + (\nu + 6)^2 \eta_n, && n = 0, 1, 2, \ldots, m - 3 \\
&\zeta_{n+1}(\nu) = \eta_{n+1} + (\nu + 8)^2 \zeta_n, && n = 0, 1, 2, \ldots, m - 4 \\
&\quad\vdots && \quad\vdots
\end{aligned} \tag{11}$$

If α and $\{\beta\}$ are given, it is straightforward to find the nonzero (nontrivial) solutions to this equation system.

The same procedure can in principle be used to solve the more general equation

$$u_t + \sum_{n=1}^{m} \gamma_n(u) u_{(2n-1)x} = 0 \tag{12}$$

where $\gamma_i(u)$ ($i = 1, 2, 3, \ldots, m$) are polynomials of u. The counterpart equality of equation (7) is in the form

$$-c + \sum_{n=1}^{m} \gamma_n(u) f_n(u) \equiv 0 \tag{13}$$

where $f_i(u)$ ($i = 1, 2, 3, \ldots, m$) are defined by equation (6). Once $\gamma_i(u)$ ($i = 1, 2, 3, \ldots, m$) are given, the solutions, in the form of equation (5), can be straightforwardly obtained.

With a series of straightforward algebraic calculations, it is possible to obtain solutions to higher-order generalized KdV equations. Furthermore, the one-dimensional solution can be generalized to two- and higher-dimensional solutions by a standard technique (Drazin and Johnson, 1989).

In summary, using a proper ansatz equation, we have obtained a class of the exact solitary wave solutions to the generalized odd-order KdV equations.

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